Quantum Hamilton-Jacobi formalism and the bound state spectra

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Abstract

It is well known in classical mechanics that, the frequencies of a periodic system can be obtained rather easily through the action variable, without completely solving the equation of motion. The equivalent quantum action variable appearing in the quantum Hamilton-Jacobi formalism, can, analogously provide the energy eigenvalues of a bound state problem, without having to solve the corresponding Schrödinger equation explicitly. This elegant and useful method is elucidated here in the context of some known and not so well known solvable potentials. It is also shown, how this method provides an understanding, as to why approximate quantization schemes such as ordinary and supersymmetric WKB, can give exact answers for certain potentials.

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I. INTRODUCTION

In classical mechanics, the Hamilton-Jacobi (H-J) theory is a well developed theory and provides an independent and often useful route for solving dynamical equations.[1] In particular, for periodic motion, the action variable enables one to obtain the frequencies of a given system directly without having to solve the equations of motion completely. The quantum H-J theory has also been studied since the inception of quantum mechanics.[2] It has been recently shown that, analogous to the classical periodic systems, the quantum action variable can be profitably employed to arrive at the energy eigenvalues for potential problems, without obtaining the corresponding wave functions.[3,4] This is to be contrasted with the standard procedure to tackle bound state problems, where, the Schrödinger equation is solved both for the eigenvalues and eigenfunctions.

The whole approach of the quantum H-J theory to the potential problems is quite elegant, requiring only some knowledge of complex variables. Keeping in mind that a student with this background will be able to appreciate it, we have made this article quite pedagogical and self-contained. In Sec. II, we briefly outline the quantum H-J formalism and its connection with the Schrödinger equation and work out the familiar harmonic oscillator example explicitly. Section III is devoted to the study of some known and not so well known potential problems to elucidate the power of this method. In Sec. IV, we show why approximate quantization schemes such as, WKB and supersymmetric (SUSY) WKB, give exact answers for certain potentials. We end this paper with some discussions and concluding remarks. For convenience, a table giving relevant information about the potential problems has also been included.

II. QUANTUM HAMILTON-JACOBI FORMALISM

Quantum H-J formalism has been developed along the lines of the classical H-J theory since the beginning of quantum mechanics, by the pioneers of the field.[2] In fact, this approach was christened as the "Royal road to quantization" in the early days of quantum mechanics.[5] In 1983, Leacock and Padgett[3,4] demonstrated that the quantum H-J formalism can yield the *exact* eigenvalues for the potential problems provided boundary conditions are applied judiciously. In what follows, after elaborating on the connection of this method with the standard text book approach to bound state problems, we solve the harmonic oscillator potential, as an illustration.

In the conventional approach to non-relativistic stationary state problems, one solves the Schrödinger equation

$$\hat{H}\psi = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi = E\psi \quad , \tag{1}$$

for the eigenvalues and eigenfunctions. In the quantum H-J formalism, the postulated quantum H-J equation,

$$\begin{aligned} \frac{\hbar}{i}\frac{\partial^2 W(x,E)}{\partial x^2} + \left(\frac{\partial W(x,E)}{\partial x}\right)^2 = \frac{\hbar}{i}\frac{\partial p(x,E)}{\partial x} + p^2(x,E) &= 2m(E - V(x)) \\ &\equiv p_c^2(x,E) \quad , \end{aligned} \tag{2}$$

replaces the Schrödinger equation as the dynamical equation.

Here, $W(x,E)$ is the quantum characteristic function and

$$p(x,E) = \frac{\partial W(x,E)}{\partial x} \quad , \tag{3}$$

is the quantum momentum function (QMF) and $p_c(x,E)$ is defined to be the classical momentum function:

$$p(x,E) \stackrel{\hbar\to 0}{\rightarrow} p_c(x,E) \quad . \tag{4}$$

This can be thought of as a manifestation of the *correspondence principle* or as a *boundary condition* on the QMF. As will be seen explicitly later, the above condition helps in determining $p(x,E)$ unambiguously. The quantum characteristic function $W(x,E)$ is related to the energy eigenvectors in the coordinate representation as,

$$\psi(x,E) = < x|E > = e^{\frac{iW(x,E)}{\hbar}} \quad , \tag{5}$$

and in the same representation,

$$\begin{aligned} < x|\hat{p}|E > = -i\hbar\frac{\partial}{\partial x} < x|E > &= \frac{\partial W}{\partial x} < x|E > \\ &= p(x,E) < x|E > \quad . \end{aligned} \tag{6}$$

Thus one gets,

$$p(x,E) = \frac{\hbar}{i}\frac{1}{\psi}\frac{\partial\psi(x,E)}{\partial x} \quad . \tag{7}$$

It is straightforward to check that the Schrödinger equation goes over to the corresponding quantum H-J equation under the above substitution and vice versa.
The quantum analog of the classical action variable is defined as

$$J(E) \equiv (1/2\pi)\oint_C dx\, p(x,E) \quad . \tag{8}$$

Here, C is a counter clockwise contour in the complex $x-$plane, enclosing the real line between the classical turning points. The turning points between which the classical motion takes place, are the real values of x, for which $p_c^2(x,E)$ vanishes. The wave function is known to have nodes between the classical turning points. These nodes correspond to poles of the quantum momentum function. To see this clearly, near a zero of the wave function, located at x_0, we write,

$$\psi = (x - x_0)\phi(x) \quad . \tag{9}$$

This implies,

$$p \approx \frac{\hbar}{i}\ \frac{1}{x - x_0} + \cdots \quad . \tag{10}$$

It is thus seen that, $p(x,E)$ has a first order pole at x_0 with residue $-i\hbar$. One can also verify the correctness of Eq. (10) directly from the quantum H-J equation. Substituting Eq. (10) in Eq. (2), one sees that the contributions of these poles from $p^2(x,E)$ and $-i\hbar\partial p(x,E)/\partial x$ cancel each other only if these poles are of first order, each having the residue $-i\hbar$. The first order poles are of quantum mechanical origin and their positions are energy dependent, being the same as the zeros of the energy eigenfunction. Just as the zeros of the wave function change their positions with energy, so do the location of the corresponding poles in the QMF. These poles will be referred to as the moving poles. The quantum H-J equation shows that, $p(x,E)$ can have singularities in the complex $x-$plane, other than the moving poles on the real axis. These singular points correspond to the singular points of the potential term $V(x)$. The locations of these singularities are, quite obviously, independent of energy; these poles will be called fixed poles.

For a given energy level the quantum number 'n' equals the number of nodes of the wave function and hence it counts the number of moving poles of $p(x,E)$ inside the contour C appearing in the definition of the quantum action variable given by Eq. (8). The residue of each of these poles is $-i\hbar$. Hence, we have

$$J(E) = n\hbar \quad , \tag{11}$$

as an exact quantization condition.

Even though this approach appears similar to that of the familiar WKB scheme, it is worth pointing out that, Eq. (11), when inverted for E reproduces the *exact* quantized energy eigenvalues. Although a priori, the location and the number of the moving poles are not known, a suitable deformation of the contour in the complex plane and change of variables allows one to compute $J(E)$, for many potentials, in terms of the fixed poles whose locations and residues are known. In what follows, explicit examples are given to clarify this method.

Examples:

1. Harmonic oscillator

The quantum H-J equation for the harmonic oscillator problem with $V(x) = m\omega^2 x^2/2$ is,

$$p^2 + \frac{\hbar}{i}\frac{\partial p(x,E)}{\partial x} = 2m(E - \frac{m\omega^2 x^2}{2}) \equiv p_c^2 \quad . \tag{12}$$

The turning points, determined from $p_c^2(x,E) = 0$, are $-x_1 = x_2 = +\sqrt{2E/(m\omega^2)}$. The quantization condition is given by

$$J(E) = (1/2\pi)\oint_C p(x,E)dx = n\hbar \quad . \tag{13}$$

Here C is the contour enclosing the moving poles between the two turning points x_1 and x_2 (see Fig. 1). Noticing that, there is only one fixed pole of $p(x,E)$ at $x \to \infty$, to evaluate $J(E)$, one considers an integral I_{Γ_R} over a circular contour Γ_R having radius R and oriented in the anti-clockwise direction. The QMF has no singular points between Γ_R and C. Hence, for this case, $J(E)$ coincides with I_{Γ_R}:

$$I_{\Gamma_R} = J(E) \quad . \tag{14}$$

For the evaluation of the contour integral I_{Γ_R}, one makes a change of variable $x = 1/y$ to get,

$$\begin{aligned} I_{\Gamma_R} &= (1/2\pi)\oint_{\Gamma_R} dx p(x,E) \\ &= (1/2\pi)\oint_{\gamma_0} dy \tilde{p}(y,E)/y^2 \quad . \end{aligned} \tag{15}$$

Here, $\tilde{p}(y,E) = p(1/y,E)$ and the counter clockwise contour γ_0 encloses only one singular point in the y plane, i.e., the pole at $y = 0$. The corresponding contour integral can be straightforwardly calculated. Note that there is no negative sign before the integral; the direction of the contour changes sense under this mapping, which is compensated by the negative sign coming from the integration measure. In

this example $J(E)$ and I_{Γ_R} are equal, though the relation between $J(E)$ and I_{Γ_R} will change from one example to another, the method of computing I_{Γ_R} remains the same for all examples.

The quantum H-J equation written in the y variable becomes

$$\tilde{p}^2(y,E) + i\hbar y^2 \frac{\partial \tilde{p}(y,E)}{\partial y} = 2m(E - \frac{m\omega^2}{2y^2}) = \tilde{p}_c^2 \ . \tag{16}$$

To calculate the contribution of the pole at $y=0$, $\tilde{p}(y,E)$ is expanded in a Laurent series as,

$$\tilde{p}(y,E) = \sum_{n=0}^{\infty} a_n y^n + \sum_{q=1}^{k} \frac{b_q}{y^q} \ . \tag{17}$$

Making use of the above expansion of $\tilde{p}(y,E)$ in Eq. (15), one notices that, the only non-vanishing contribution comes from the coefficient a_1 of the linear term in y.

In the next step, substituting $\tilde{p}(y,E)$ in the quantum H-J equation and comparing the left and right hand sides, it is found that, $b_q = 0$ for $q > 1$. On equating the coefficients of the different powers of y, we have,

$$b_1^2 = -m^2\omega^2 \ , \tag{18}$$

$$2a_0 b_1 = 0 \ , \tag{19}$$

$$-i\hbar b_1 + 2a_1 b_1 + a_0^2 = 2mE \ . \tag{20}$$

From Eq. (18), one finds $b_1 = \pm im\omega$. This ambiguity in sign for b_1 can be removed, if we apply the boundary condition given by Eq. (4). In the convention followed here, the classical momentum function is defined such that $p_c(x,E) = +i|p_c|$ on the positive real axis.[4] In the limit $y \to 0, (x \to \infty)$, $p_c \approx im\omega/y$ and therefore from Eq. (17) it follows that $b_1 = im\omega$. From Eq. (19), we then have $a_0 = 0$. Substituting the value of b_1 in Eq. (20) one gets $a_1 = (2E - \hbar\omega)/(2i\omega)$. Plugging Eq. (17) in Eq. (15) and noting Eq. (14) gives,

$$J(E) = I_{\Gamma_R} = ia_1 = \frac{2E - \hbar\omega}{2\omega} \ . \tag{21}$$

Thus the quantization condition $J(E) = n\hbar$ when inverted for E, gives

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \ . \tag{22}$$

2. *Harmonic oscillator on half line*

The quantum H-J equation for this case is obviously the same as that of the previous example, apart from the condition that the potential is ∞ at $x = 0$. This forces us to assume a fixed pole for $p(x, E)$ in the complex x plane at $x = 0$ and this also serves as one of the turning points. The other turning point is located at $x_2 = \sqrt{2E/(m\omega^2)}$. We have,

$$J(E) = (1/2\pi) \oint_C p(x)dx \quad , \tag{23}$$

where C is the contour enclosing the moving poles between zero and $\sqrt{2E/(m\omega^2)}$. To evaluate $J(E)$, we consider a contour integral I_{Γ_R} over a circle of radius R in the complex plane; R is taken large enough to enclose all the singular points of the QMF inside it. This contour integral,

$$I_{\Gamma_R} \equiv (1/2\pi) \oint_{\Gamma_R} p(x)dx \quad , \tag{24}$$

gets contributions from the singular points of $p(x, E)$ inside Γ_R. These are, (i) fixed pole at $x = 0$, (ii) moving poles between 0 and $x_2 = \sqrt{2E/(m\omega^2)}$ and (iii) moving poles on the negative real axis between 0 and $x_1 = -\sqrt{2E/(m\omega^2)}$. The poles on the negative real axis arise due to the symmetry $x \to -x$ of the problem. Hence,

$$I_{\Gamma_R} = J(E) + I_{C_1} + I_{\gamma_1} \quad , \tag{25}$$

where I_{γ_1} is the contour integral for the contour γ_1 that takes into account the additional pole at $x = 0$ and I_{Γ_R} is the contour integral for the contour Γ_R enclosing all the moving and fixed poles of $p(x, E)$. I_{C_1} is the contour integral for the counter clockwise contour C_1, enclosing the moving poles on the negative side of the real

axis (see Fig. 2). However, under $x \to -x$, the turning points x_1 and x_2 and the moving poles get interchanged. Thus,

$$I_{C_1} = J(E) \quad . \tag{26}$$

Substituting the above in Eq. (25) we have,

$$I_{\Gamma_R} = 2J(E) + I_{\gamma_1} \quad . \tag{27}$$

Observing that, it is only the b_1/x term in the Laurent expansion of $p(x, E)$ that contributes to the contour integral I_{γ_1}, the relevant relation involving b_1 following from the quantum H-J equation is,

$$b_1^2 - \frac{\hbar}{i} b_1 = 0 \quad . \tag{28}$$

From the two solutions $b_1 = 0; -i\hbar$, the former is discarded, since it does not give rise to the required singularity at $x = 0$. Hence, using the second value,

$$I_{\gamma_1} = \hbar \quad . \tag{29}$$

I_{Γ_R} is evaluated as in the previous section, and is found to be exactly the same. We now substitute the values of I_{γ_1} and I_{Γ_R} from Eqs. (29) and (21) in Eq. (27). The quantization rule $J(E) = n\hbar$, when inverted for E, gives the energy eigenvalue equation;

$$E = \left((2n + 1) + \frac{1}{2}\right) \hbar\omega \quad . \tag{30}$$

III. OTHER SOLVABLE EXAMPLES

In this section, the quantum H-J method will be applied to a set of trigonometric and hyperbolic potentials.[6] These potentials have attracted considerable attention in the recent literature due to the fact that they can be solved algebraically using the techniques of SUSY quantum mechanics.[7] As will be seen explicitly later, SUSY

shows an alternate way of selecting the correct solutions of the quantum H-J equation. Suitable exponential mappings are used in the analyses of the trigonometric and hyperbolic potentials. Here, the example of the Eckart potential will be worked out in detail and the relevant information about the rest of the potentials will be given in Table I. We also work out the infinite square well potential in this section which has been deliberately tackled at the end because of its nontrivial nature in context of the present formalism.

1. Eckart potential

The Eckart potential, by suitably adjusting the ground state energy, can be written in the form,

$$V(x) = \omega^2(x) - \frac{\hbar}{\sqrt{2m}} \frac{\partial \omega(x)}{\partial x} \quad ,$$

where

$$\omega(x) = -A \coth \alpha x + B/A \quad , \tag{31}$$

is known as the superpotential in the literature.[7] Given the ground state wave function ψ_0, in SUSY quantum mechanics,

$$\omega(x) = -\frac{\hbar}{\sqrt{2m}} \frac{1}{\psi_0} \frac{\partial \psi_0}{\partial x} \quad . \tag{32}$$

The QMF, $p(x,E)$, becomes equal to $i\sqrt{2m}\omega(x)$ for $E = 0$. The quantum H-J equation for the Eckart potential is given as,

$$p^2(x,E) + \frac{\hbar}{i} \frac{\partial p(x,E)}{\partial x} = 2m \left(E - A^2 - \frac{B^2}{A^2} - A(A - \frac{\alpha\hbar}{\sqrt{2m}}) \mathrm{cosech}^2 \alpha x + 2B \coth \alpha x \right) \quad , \tag{33}$$

where x lies on the half line. To simplify the analysis, we use the mapping $y = \exp(\alpha x)$ and the corresponding equations (31) and (33) become

$$\omega(y) = -A \frac{y^2+1}{y^2-1} + \frac{B}{A} \quad , \tag{34}$$

and

$$\tilde{p}^2(y,E) + \frac{\hbar\alpha y}{i}\frac{\partial\tilde{p}(y,E)}{\partial y} = 2m\left(E - A^2 - \frac{B^2}{A^2} - \frac{4A(A-\frac{\alpha\hbar}{\sqrt{2m}})y^2}{(y^2-1)^2} + \frac{2B(y^2+1)}{(y^2-1)}\right) \quad , \tag{35}$$

respectively. Here $\tilde{p}(y,E) = p(\log y/\alpha, E)$. The right hand side of Eq. (35) equated to zero has four solutions, out of which two turning points are in the physical regime. The quantization condition in the y variable takes the form,

$$J(E) = \frac{1}{2\pi\alpha}\oint_{C_1}\frac{dy}{y}\tilde{p}(y,E) = n\hbar \quad . \tag{36}$$

It is clear from Eqs. (35) and (36), that the integrand has singularities at $y = 0, \ \pm 1$. As before, we shall now consider a circle Γ_R of radius R, which is such that all the singular points of the integrand in Eq. (36) are enclosed. Then

$$I_{\Gamma_R} = I_{C_1} + I_{C_2} + I_{\gamma_1} + I_{\gamma_2} + I_{\gamma_3} \quad . \tag{37}$$

where $I_{C_1} \equiv J(E)$ and I_{C_2} are the integrals along the counter clockwise contours C_1 and C_2 enclosing the classical and non-classical turning points respectively. I_{γ_1}, I_{γ_2} and I_{γ_3} are the integrals along contours γ_1, γ_2 and γ_3 which encloses the singular points at $y = 1, y = -1$ and $y = 0$ respectively (see Fig. 3). It may be noticed that, the symmetry $y \to -y$ in Eq. (35) interchanges the turning points in the non-classical region with those in the classical region. Thus,

$$I_{C_1} = I_{C_2} \quad . \tag{38}$$

Therefore, from Eq. (37), we have

$$2J(E) = I_{\Gamma_R} - I_{\gamma_1} - I_{\gamma_2} - I_{\gamma_3} \quad . \tag{39}$$

To find the contribution for the pole at $y = 1$, one expands

$$\tilde{p}(y,E) = \frac{b_1}{(y-1)} + a_0 + a_1(y-1) + \cdots \quad , \tag{40}$$

and substitutes the same into Eq. (35). Comparing the coefficients of $1/(y-1)^2$ on both sides gives,

$$b_1 = \frac{-i\alpha\hbar \pm i(\alpha\hbar - 2\sqrt{2m}A)}{2} \quad . \tag{41}$$

Similarly for the pole at $y = -1$ an expansion in powers of $(y+1)$ is sought in $\tilde{p}(y, E)$ and we have

$$b_1' = \frac{i\alpha\hbar \pm i(\alpha\hbar - 2\sqrt{2m}A)}{2} \quad , \tag{42}$$

where b_1' is the coefficient of the $1/(y+1)$ term in the above expansion. To find the residue of the integrand at the pole at $y = 0$, we expand $\tilde{p}(y, E)$ as

$$\tilde{p}(y, E) = a_0 + a_1 y + \cdots \quad , \tag{43}$$

and comparing the coefficient of the constant term gives

$$a_0^2 = 2m(E - A^2 - \frac{B^2}{A^2} - 2B) \quad . \tag{44}$$

For the calculation of I_{Γ_R}, one more change of variable in the form of $y = 1/z$ is sought, so that the singularity at $y \to \infty$ is mapped to the one at $z = 0$. Proceeding as before, the coefficient of the constant term in the expansion of $\tilde{\tilde{p}}(z, E)$ in powers of z is,

$$a_0'^2 = 2m(E - A^2 - \frac{B^2}{A^2} + 2B) \quad , \tag{45}$$

where $\tilde{\tilde{p}}(z, E) = \tilde{p}(1/z, E)$. To find the spectrum for the Eckart potential, one has to choose correct signs for the above coefficients, by appropriate boundary conditions. The procedure as originally suggested by Leacock and Padgett is a bit complicated to implement. Instead, we adopt an alternate method, where a similar expansion of the superpotential $\omega(x)$ at the given poles is compared with the $E \to 0$ limit of the above coefficients, there by fixing the correct sign. This is because Eq. (32) implies that $p(x, E) = i\sqrt{2m}\omega(x)$ at zero energy. Therefore from Eqs. (41), (42), (44) and

(45), we have,

$$\alpha\, I_{\gamma_1} = \sqrt{2m}A \quad , \tag{46}$$

$$\alpha\, I_{\gamma_2} = \sqrt{2m}A \quad , \tag{47}$$

$$\alpha\, I_{\gamma_3} = -i\sqrt{2m(E - A^2 - \frac{B^2}{A^2} - 2B)} \quad , \tag{48}$$

$$\alpha\, I_{\Gamma_R} = \sqrt{2m(E - A^2 - \frac{B^2}{A^2} + 2B)} \quad . \tag{49}$$

Taking into account the contribution of the two identical contours enclosing the moving poles in the complex $y-$plane, we find

$$I_{\Gamma_R} - \sum_{p=1}^{3} I_{\gamma_p} = 2n\hbar \quad . \tag{50}$$

Solving for E:

$$E_n = A^2 + \frac{B^2}{A^2} - \frac{B^2}{(\frac{n\alpha\hbar}{\sqrt{2m}} + A)^2} - (\frac{n\alpha\hbar}{\sqrt{2m}} + A)^2 \quad . \tag{51}$$

The calculation of eigenvalues for other SUSY potentials proceeds along the same lines and the results are summarised in Table I.

2. The square well potential

Square well potential of width L is the simplest example of one-dimensional motion where a particle of mass m experiences a potential,

$$\begin{aligned} V(x) &= 0 \quad \text{for } 0 < x < L \quad , \\ &= \infty \quad \text{for } x \leq 0 \text{ and } x \geq L \quad . \end{aligned} \tag{52}$$

The nodes of the eigenfunctions and hence the moving poles of $p(x,E)$ are located between $x = 0$ and L. Let C be a rectangular contour enclosing all the moving poles (see Fig.4a). Then the quantization condition is given by

$$J(E) = \frac{1}{2\pi}\oint_C p(x,E)dx = n\hbar \quad , \tag{53}$$

where the QMF obeys the quantum H-J equation,

$$p^2(x,E) - i\hbar\frac{\partial p(x,E)}{\partial x} = 2mE \quad . \tag{54}$$

We now use a mapping $z = \exp((2\pi i x)/L)$; the contour C in the $x-$plane (Fig. 4a) is mapped into the contour $PAQRBSP$ in the $z-$plane . The moving poles get mapped on the middle arc of unit radius (Fig. 4b). The quantization condition is now given as

$$J(E) = \frac{L}{4i\pi^2}\oint_{C_1}\frac{\tilde{p}(z,E)dz}{z} = n\hbar \quad , \tag{55}$$

where C_1 is the contour $PAQRBSP$ of Fig. 4b. The quantum H-J equation written in terms of the new variable z is,

$$\tilde{p}^2(z.E) + \frac{2\pi\hbar z}{L}\frac{\partial\tilde{p}(z,E)}{\partial z} = 2mE \quad , \tag{56}$$

where $\tilde{p}(z,E) = p(L\log(z)/(2\pi i), E)$. The boundary condition that, the wave function vanishes at $x = 0$ and $x = L$ gives rise to a pole in $\tilde{p}(z,E)$ at $z = 1$. Let γ and Γ be the inner and outer *full* circles of radii OA and OB respectively, both taken in the anti-clockwise direction. The integral in Eq. (55) can be written in terms of integrals over Γ, γ and the contour $P'S'R'Q'P'$ enclosing the pole at $z = 1$. Thus, we get

$$J(E) = \frac{L}{4i\pi^2}\left(\oint_{\Gamma}\frac{\tilde{p}(z,E)dz}{z} - \oint_{\gamma}\frac{\tilde{p}(z,E)dz}{z} - \oint_{P'S'R'Q'P'}\frac{\tilde{p}(z,E)dz}{z}\right) \quad . \tag{57}$$

The first integral is computed by changing variables from z to $y = 1/z$, as was done earlier for Γ_R. The last two integrals in the above expressions are calculated as usual by doing the Laurent expansion of $\tilde{p}(z,E)$ in powers of z and $z-1$ respectively and we have,

$$\oint_{\Gamma}\frac{\tilde{p}(z,E)dz}{z} \quad = \quad -2\pi i\sqrt{2mE} \tag{58}$$

$$\oint_{\gamma} \frac{\tilde{p}(z,E)dz}{z} = 2\pi i\sqrt{2mE} \tag{59}$$

$$\oint_{P'S'R'Q'P'} \frac{\tilde{p}(z,E)dz}{z} = \frac{4\pi^2 i\hbar}{L} \quad . \tag{60}$$

Substituting the above in Eq. (57), and solving for E we get,

$$E = \left(\frac{\pi^2\hbar^2}{2mL^2}\right)(n+1)^2 \quad , \quad n = 0, 1, 2, \cdots \quad . \tag{61}$$

IV. EXACTNESS OF ORDINARY AND SUSY WKB APPROXIMATION SCHEMES

Semiclassical quantization schemes like WKB have been very useful since the early days of quantum mechanics, in getting the approximate spectra of bound state problems. Interestingly, for certain potentials, these approximation schemes give exact answers. It is well known that WKB quantization condition,

$$\int_{x_1}^{x_2} \sqrt{2m(E - V(x))}dx = (n + \frac{1}{2})\pi\hbar \quad , \tag{62}$$

reproduces the exact eigenvalues for the harmonic oscillator potential. Here, x_1 and x_2 are the two classical turning points with $x_1 < x_2$ and n takes positive integral values.

With the advent of SUSY quantum mechanics it was found that, for a potential $V(x) = \omega^2(x) - \frac{\hbar}{\sqrt{2m}}\frac{\partial\omega(x)}{\partial x}$, for which the ground state energy is zero; a variation of the WKB approximation,

$$\int_{x_1}^{x_2} \sqrt{2m(E - \omega^2(x))} = n\hbar\pi; \qquad n = 0, 1, 2, \ldots \ , \tag{63}$$

gives exact eigenvalues for a large class of potentials.[8–10] Here x_1 and x_2 are solutions of $E - \omega^2 = 0$ with $x_1 < x_2$. This is the well known SUSY WKB approximation scheme. Since quantum H-J formalism gives exact results, and is very similar to these schemes, it can be used to gain an understanding as to why SUSY WKB

approximation reproduces the exact answers for these potentials.

For the harmonic oscillator, where WKB is exact, it can be shown that WKB and SUSY WKB approximations are identical. Therefore, we concentrate only on the SUSY WKB scheme here. In order to compare SUSY WKB with the quantum H-J quantization condition, the line integral will be converted to a suitable contour integration over the complex $x-$plane, with a counter clockwise contour C, enclosing the turning points x_1 and x_2 .[11] Writing,

$$\frac{1}{\pi}\int_{x_1}^{x_2}\sqrt{2m(E-\omega^2)}dx = \frac{1}{2\pi}\oint_C \sqrt{2m(E-\omega^2)}dx = n\hbar \quad , \tag{64}$$

the contour integral in Eq. (64) can be evaluated, for all the potentials tabulated in Table I. Amazingly, for all these cases, the singularity structure of $\sqrt{2m(E-\omega^2)}$, other than the branch cuts, matches exactly with that of the fixed poles of $p(x,E)$ in the quantum H-J formalism; the location of the fixed poles and the corresponding residues are identical for both the cases.[12] As has been seen earlier, these poles and their residues determine the eigenspectra completely. Hence, it is not surprising that SUSY WKB gives exact answers for these potentials.

V. CONCLUSION:

We have explicitly worked out the eigenvalues for several solvable potentials in one dimension, using the quantum H-J method. Table I contains all the relevant details about the steps involved in the calculation. Interestingly, the fixed poles of the quantum momentum function, which are not of quantum mechanical origin, in conjunction with the boundary conditions, completely determine the spectra for these examples. The main effort involved in use of this scheme, lies in selecting the correct roots for the residues needed. This was done, first, by using the boundary condition on the QMF $p(x,E)$, viz., $p \rightarrow p_c$ in the limit $\hbar \rightarrow 0$. It was then shown that for the SUSY potentials, comparison of the answers obtained from the quantum H-J for $E=0$ with that obtained from the superpotential also reproduced the correct

roots.

For the potentials, where certain approximate quantization schemes are exact, this approach has provided some useful insight: It is the similarity of the singularity structure of $p(x, E)$ and $\sqrt{2m(E-\omega^2)}$ in the non-classical regions of the $x-$plane, namely the matching of the poles and the residues that is responsible for this exactness.

Although, we have not dealt with them here, for the non-exact potentials, it is the inability to deform the contour appropriately because of the presence of other poles and branch cuts in the complex $x-$plane that prevents an exact solution in this approach.[12] Since, exactly solvable potentials are few and far between, this method can be potentially useful to construct new ones.

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Table I : Hyperbolic and trigonometric potentials.

The mapping used for hyperbolic potentials is $y = \exp(\alpha x)$, while

for the trigonometric ones $y = \exp(i\alpha x)$ is being used; α is real and positive.

Name of potential	Potential	Fixed poles at	αI_{γ_p}	Eigenvalue
Scarf II (hyperbolic)	$A^2 + (B^2 - A^2 - \frac{A\alpha\hbar}{\sqrt{2m}})\times$ $\operatorname{sech}^2 \alpha x + B(2A + \frac{\alpha\hbar}{\sqrt{2m}})\times$ $\operatorname{sech} \alpha x \, \tanh \alpha x$	0 i $-i$ ∞	$-i\sqrt{2m(E-A^2)}$ $\sqrt{2m}(iB - A)$ $-\sqrt{2m}(iB + A)$ $i\sqrt{2m(E-A^2)}$	$A^2 - (A - \frac{n\alpha\hbar}{\sqrt{2m}})^2$
Rosen - Morse II (hyperbolic)	$A^2 + \frac{B^2}{A^2}$ $-A(A + \frac{\alpha\hbar}{\sqrt{2m}}) \operatorname{sech}^2 \alpha x$ $+2B \tanh \alpha x$	0 i $-i$ ∞	$-i\sqrt{2m(E - A^2 - \frac{B^2}{A^2} + 2B)}$ $-\sqrt{2m}A$ $-\sqrt{2m}A$ $i\sqrt{2m(E - A^2 - \frac{B^2}{A^2} - 2B)}$	$A^2 + B^2/A^2$ $-(A - \frac{n\alpha\hbar}{\sqrt{2m}})^2$ $-B^2/(A - \frac{n\alpha\hbar}{\sqrt{2m}})^2$
Generalised Poschl- Teller (hyperbolic)	$A^2 + (B^2 + A^2 + \frac{A\alpha\hbar}{\sqrt{2m}})\times$ $\operatorname{cosech}^2 \alpha x - B(2A + \frac{\alpha\hbar}{\sqrt{2m}})\times$ $\coth \alpha x \operatorname{cosech} \alpha x$ $(x \geq 0)$	0 1 -1 ∞	$-i\sqrt{2m(E-A^2)}$ $-\sqrt{2m}(A - B)$ $-\sqrt{2m}(A + B)$ $i\sqrt{2m(E-A^2)}$	$A^2 - (A - \frac{n\alpha\hbar}{\sqrt{2m}})^2$
Scarf I (trig- onometric)	$-A^2 + (A^2 + B^2 + \frac{A\alpha\hbar}{\sqrt{2m}})\times$ $\sec^2 \alpha x - B(2A + \frac{\alpha\hbar}{\sqrt{2m}})\times$ $\sec \alpha x \tan \alpha x$ $(-\pi/2 \leq \alpha x \leq \pi/2)$	0 i $-i$ ∞	$\sqrt{2m(E+A^2)}$ $-\sqrt{2m}(A - B)$ $-\sqrt{2m}(A + B)$ $-\sqrt{2m(E+A^2)}$	$(A + \frac{n\alpha\hbar}{\sqrt{2m}})^2 - A^2$
Rosen- Morse-I (trig- onometric)	$A(A - \frac{\alpha\hbar}{\sqrt{2m}}) \operatorname{cosec}^2 \alpha x$ $-A^2 + \frac{B^2}{A^2}$ $+2B \cot \alpha x$ $(0 \leq \alpha x \leq \pi)$	0 1 -1 ∞	$-\sqrt{2m(E + A^2 - \frac{B^2}{A^2} + 2iB)}$ $\sqrt{2m}A$ $\sqrt{2m}A$ $\sqrt{2m(E + A^2 - \frac{B^2}{A^2} - 2iB)}$	$A^2 + B^2/A^2$ $-(A + \frac{n\alpha\hbar}{\sqrt{2m}})^2$ $-B^2/(A + \frac{n\alpha\hbar}{\sqrt{2m}})^2$

Figure Captions

1. Figure 1. Contour for the harmonic oscillator problem.

2. Figure 2. Contour for the harmonic oscillator problem on half line.

3. Figure 3. Contour for the Eckart potential problem, using the mapping $y = \exp(\alpha x)$.

4. Figure 4a. Contour for the square well problem, in the $x-$plane.

5. Figure 4b. Contour for the square well problem, in the $z-$plane.